Last time: least squares solutions to an inconsistent system $Ax = b$ is

$$A x^* = \mathrm{proj}_{\mathrm{col}(A)}(b)$$

$$\Updownarrow$$

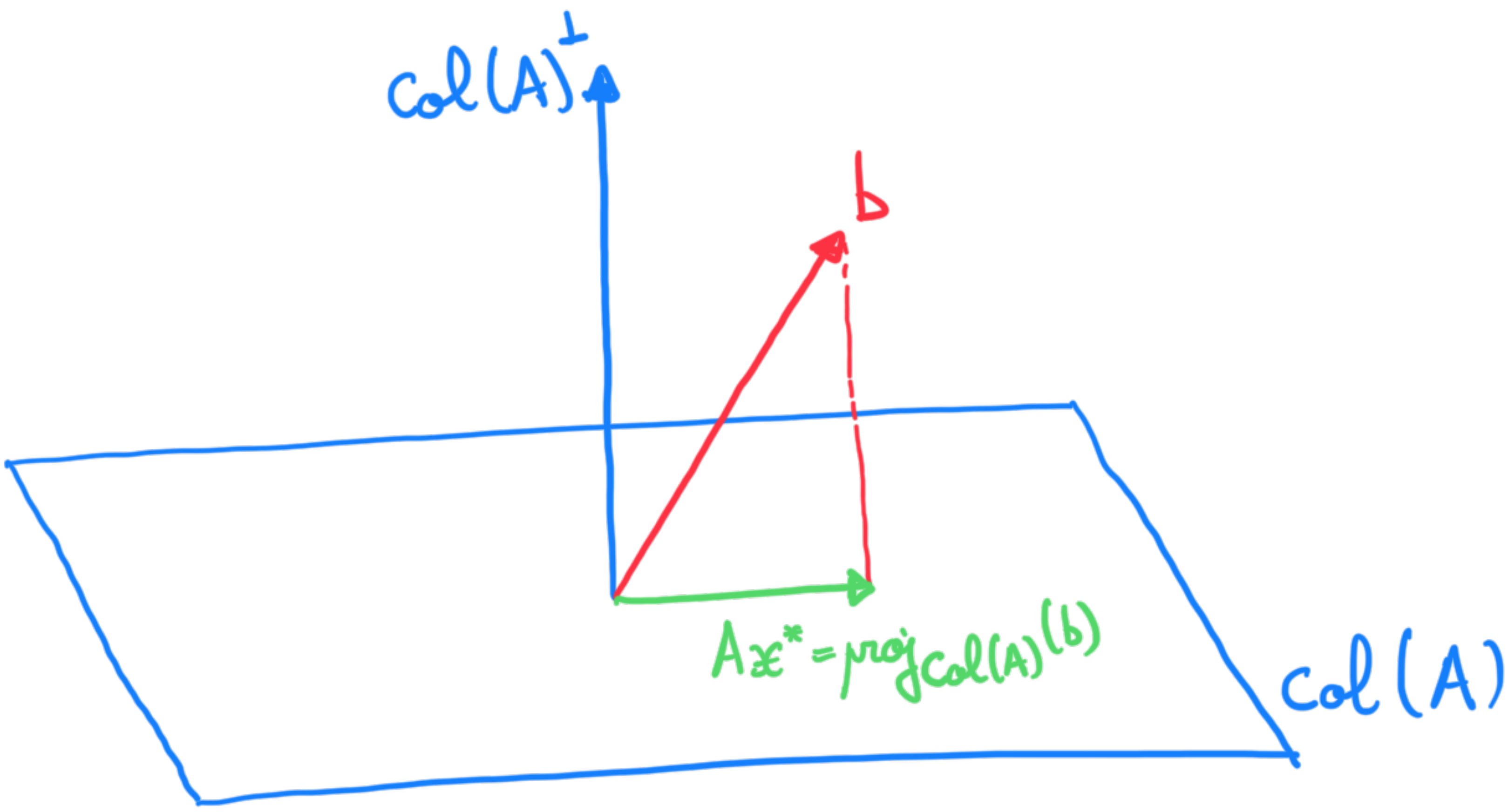
$$A^T A x^* = A^T b$$

unique solution if A^TA is invertible

$$\Updownarrow$$

$$x^* = R^{-1} Q^T b$$

$A = QR$ has linearly independent columns



Today: "We had diagonalization yet!

Today: "We had diagonalization, yes?

But what about orthogonal diagonalization?"

$$B = PDP^{-1}$$

$P = $ (eigenvectors of B)

$\Updownarrow$

$\exists$ basis of eigenvectors

$$\boxed{B = UDU^{-1}}$$

$U = $ (orthonormal eigenvectors of B)

$\Updownarrow$

$\exists$ orthonormal basis of eigenvectors

All matrices in this lecture are $n \times n$

Recall: $U \in \mathbb{R}^{n \times n}$ has orthonormal columns

$\Updownarrow$

$U^T = U^{-1}$ (orthogonal matrix)

DEF 25.1 : a square matrix B is called **orthogonally diagonalizable** if $\exists$ square D, U s.t.

$$\boxed{B = UDU^{-1}}$$

where D is diagonal and U is orthogonal (i.e. $U^{-1} = U^T$)

What is a necessary condition on B so that it's orthogonally diagonalizable?

B has to be symmetric, i.e. $B = B^T$

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots \\ b_{21} & b_{22} & b_{23} & \cdots \\ b_{31} & b_{32} & b_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & b_{31} & \cdots \\ b_{12} & b_{22} & b_{32} & \cdots \\ b_{13} & b_{23} & b_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Ex: $\begin{pmatrix} 2 & 9 \\ 9 & 5 \end{pmatrix}$ is symmetric, but $\begin{pmatrix} 0 & 4 \\ 5 & 0 \end{pmatrix}$ is not

$\begin{pmatrix} -1 & 8 & 7 \\ 8 & 5 & -2 \\ 7 & -2 & 4 \end{pmatrix}$ is symmetric, but $\begin{pmatrix} -1 & 8 & 7 \\ 8 & 5 & -3 \\ 7 & -2 & 4 \end{pmatrix}$ is not

(being symmetric only applies to square matrices)

(anti-symmetric matrix : $B^T = -B$, e.g. $\begin{pmatrix} 0 & 3 & 7 \\ -3 & 0 & 4 \\ -7 & -4 & 0 \end{pmatrix}$)

THM 25.2: if a square matrix B is orthogonally diagonalizable, then B is symmetric.

Proof: $B = UDU^{-1}$ where $U^T = U^{-1}$

$$U^{T,-1} = U = U^{-1,T}$$

transpose

$$B^T = U^{-1,T} D^T U^T$$

because D is diagonal hence symmetric $\begin{pmatrix} d_1 & 0 & 0 & \cdots \\ 0 & d_2 & 0 & \cdots \\ 0 & 0 & d_3 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$

$$= U D U^{-1}$$

$$= B$$

B is symmetric

$\square$

Goal: prove the converse, i.e. B symmetric

$\Downarrow$

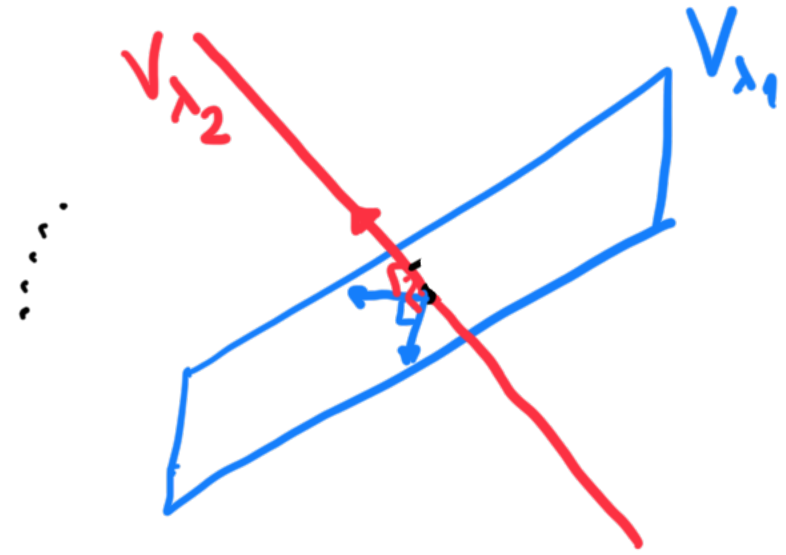
B orthogonally diagonalizable

real $n \times n$

THM 25.3: B is orthogonally diagonalizable if the following facts all hold

(1) all eigenvalues $\lambda_1, \ldots, \lambda_n$ are real

(2) geom. mult. λ_i = alg. mult. λ_i, for all i



(3) $V_{\lambda_i} \perp V_{\lambda_j}$ for all $\lambda_i \neq \lambda_j$

THM 25.4 : if B is symmetric, conditions (1)–(3) are satisfied $\Rightarrow$ B is orthogonally diagonalizable

Lemma: for any matrix A and any vectors v, w

$$\underbrace{A}_{m\times n}\underbrace{v}_{n\times 1} \cdot \underbrace{w}_{m\times 1} = \underbrace{v}_{n\times 1} \cdot \underbrace{A^T}_{n\times m}\underbrace{w}_{m\times 1}$$

(proof: $Av \cdot w = (Av)^T w = v^T A^T w = v^T(A^T w) = v \cdot A^T w$)

Proof : (1) any eigenvalue λ of a symmetric matrix B is real

$B v = \lambda v$ for some vector v (can be complex)

$$Bv \cdot \bar{v} = \lambda v \cdot \bar{v} \quad (*)$$

$\Downarrow$ Lemma

$v = \begin{pmatrix} a+bi \\ c+di \end{pmatrix}, \quad \bar{v} = \begin{pmatrix} a-bi \\ c-di \end{pmatrix}$

$$v \cdot B^T \bar{v} = \lambda v \cdot \bar{v}$$

(B symmetric)

$$v \cdot B\bar{v} = \lambda v \cdot \bar{v}$$

$$B\bar{v} \cdot v = \lambda v \cdot \bar{v}$$

(*) conjugate

$$\bar{B}\bar{v} \cdot \bar{\bar{v}} = \bar{\lambda}\,\bar{v} \cdot \bar{\bar{v}}$$

$$B\bar{v} \cdot v = \bar{\lambda}\,\bar{v} \cdot v = \bar{\lambda}\, v \cdot \bar{v}$$

$$\lambda v \cdot \bar{v} = \bar{\lambda} v \cdot \bar{v}$$

$\Downarrow$

$\lambda = \bar{\lambda}$, i.e. λ real

(because $v \cdot \bar{v} \neq 0$)

(2) hard (beyond the scope of this class)

(3) show that distinct eigenspaces are orthogonal

$$\underset{\substack{\cup \\ v}}{V_\lambda} \perp \underset{\substack{\cup \\ w}}{V_\mu} \quad \text{if } \lambda \neq \mu$$

Need to show $v \cdot w = 0$

$Bv = \lambda v$

$Bw = \mu w$

$$\mu v \cdot w = v \cdot Bw = v \cdot B^T w \overset{\text{Lemma}}{=} Bv \cdot w = \lambda v \cdot w$$

$\Downarrow$ because $\lambda \neq 0$

$$v \cdot w = 0$$

$\square$

Ex: is $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ orthogonally diagonalizable?

$\Updownarrow$

Symmetric

Yes! Then orthogonally diagonalize it, i.e. find D, U s.t.

$$B = UDU^{-1}$$

where D is diagonal and U is orthogonal

- compute eigenvalues:

$$\chi_B(t) = \det\begin{pmatrix} t-2 & -1 \\ -1 & t-3 \end{pmatrix} = (t-2)(t-3) - 1 = t^2 - 5t + 5$$

has roots $\lambda_1 = \dfrac{5+\sqrt{5}}{2}$, $\lambda_2 = \dfrac{5-\sqrt{5}}{2}$ (both real)

$$V_{\lambda_1} = \operatorname{Ker}(B - \lambda_1 \cdot I_2) = \operatorname{Ker}\begin{pmatrix} 2-\lambda_1 & 1 \\ 1 & 3-\lambda_1 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{5}}{2} & 1 \\ 1 & \frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix}$$

subtract $\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)$ row$_1$ from row$_2$

$$\operatorname{Ker}\begin{pmatrix} -\frac{1}{2}-\frac{\sqrt{5}}{2} & 1 \\ \underbrace{1-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}_{=0} & \underbrace{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}_{=0} \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -\frac{1}{2}-\frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{pmatrix}$$

$$V_{\lambda_1} = \operatorname{span}\begin{pmatrix} 1 \\ \frac{1}{2}+\frac{\sqrt{5}}{2} \end{pmatrix} =: b_1$$

$$V_{\lambda_2} = \operatorname{span}\begin{pmatrix} 1 \\ \frac{1}{2}-\frac{\sqrt{5}}{2} \end{pmatrix} =: b_2$$

$$g_1 = \frac{b_1}{\|b_1\|} = \frac{b_1}{d_1} \qquad g_2 = \frac{b_2}{\|b_2\|} = \frac{b_2}{d_2}$$

$$\|b_1\| = \sqrt{1+\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^2} = \sqrt{1+\frac{1}{4}+\frac{5}{4}+\frac{\sqrt{5}}{2}}$$

$$= \frac{1}{2}\sqrt{10+2\sqrt{5}} = d_1$$

similarly, $\|b_2\| = \frac{1}{2}\sqrt{10-2\sqrt{5}} = d_2$

$V_{\lambda_1} \perp V_{\lambda_2}$, because

$$\begin{pmatrix} 1 \\ \frac{1}{2}+\frac{\sqrt{5}}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \frac{1}{2}-\frac{\sqrt{5}}{2} \end{pmatrix} = 1\cdot 1 + \left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\cdot\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right) = 0$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \begin{pmatrix} \frac{5+\sqrt{5}}{2} & 0 \\ 0 & \frac{5-\sqrt{5}}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d_1} & \frac{1}{d_2} \end{pmatrix}$$

$$U = \begin{pmatrix} g_1 & g_2 \end{pmatrix} = \begin{pmatrix} \cdots & \cdots \\ \frac{\frac{1}{2} + \frac{\sqrt{5}}{2}}{d_1} & \frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{d_2} \end{pmatrix}$$

$$\Downarrow$$

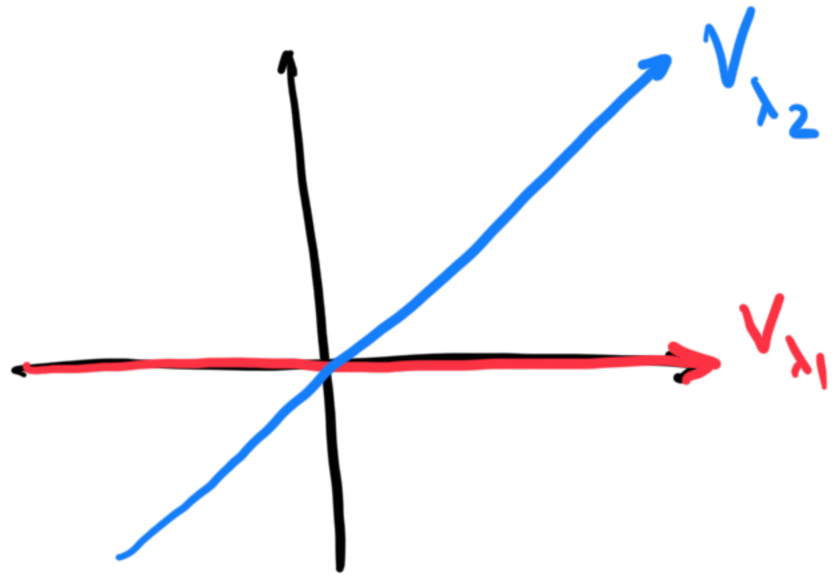
$B = UDU^{-1}$ is the orthogonal diagonalization of B

Non-ex: $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ is not symmetric, hence not orthogonally diagonalizable

$\lambda_1 = 2$ $\qquad$ $\lambda_2 = 3$

$$V_{\lambda_1} = \operatorname{span} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{=: v_1} \text{ because } \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$V_{\lambda_2} = \operatorname{Ker}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = \operatorname{span} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{=: v_2}$$



because $\begin{pmatrix}1\\0\end{pmatrix} \not\perp \begin{pmatrix}1\\1\end{pmatrix}$, $V_{\lambda_1} \not\perp V_{\lambda_2}$, so

$B = \begin{pmatrix}2 & 1\\0 & 3\end{pmatrix}$ is not orthogonally diagonalizable,

even though it is diagonalizable $B = PDP^{-1}$ with

$$D = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}, \quad P = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$$

Can G-S on the columns of P help? No, because

$v_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad b_1 = \begin{pmatrix}1\\0\end{pmatrix}$

$v_2 = \begin{pmatrix}1\\1\end{pmatrix}$ $\rightsquigarrow$ $\quad b_2 = v_2 - \text{proj}_{b_1}(v_2) = \begin{pmatrix}1\\1\end{pmatrix} - \text{proj}_{\begin{pmatrix}1\\0\end{pmatrix}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

b_2 thus obtained is no longer an eigenvector, hence useless for the purpose of orthogonal diagonalization

Ex: orthogonally diagonalize $B = \begin{pmatrix}-4 & 3 & 0\\3 & 4 & 0\\0 & 0 & 5\end{pmatrix}$

. eigenvalues $-5, 5, 5$

alg. mult 1 $\qquad$ alg. mult 2

$$\cdot\ V_{-5} = \operatorname{span}\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} =: v_1$$

$$\cdot\ V_5 = \operatorname{span}\left\{\underbrace{\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}}_{=:v_2}, \underbrace{\begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}}_{=:v_3}\right\}$$

$$v_1 \cdot v_2 = -3\cdot 1 + 1\cdot 3 + 0\cdot 1 = 0$$

$$v_1 \cdot v_3 = -3\cdot 2 + 1\cdot 6 + 0\cdot 0 = 0$$

$$v_2 \cdot v_3 = 1\cdot 2 + 3\cdot 6 + 1\cdot 0 = 20 \neq 0$$

Let $\quad b_1 = v_1$

G-S in V_5: $\quad b_2 = v_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

$$b_3 = v_3 - \operatorname{proj}_{\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}}(v_3)$$

$$= \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix} - \frac{20}{11}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$= \frac{1}{11}\begin{pmatrix} 2 \\ 6 \\ -20 \end{pmatrix}$$

sanity check: $b_2 \cdot b_3 = \frac{1}{11}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 6 \\ -20 \end{pmatrix} = \frac{1}{11}\left(1\cdot 2 + 3\cdot 6 - 1\cdot 20\right) = 0$

$$q_1 = \frac{b_1}{\|b_1\|} = \frac{1}{\sqrt{10}}\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

$$g_2 = \frac{b_2}{\|b_2\|} = \frac{1}{\sqrt{11}}\begin{pmatrix}1\\3\\1\end{pmatrix}$$

$$g_3 = \frac{b_3}{\|b_3\|} = \frac{1}{\sqrt{440}}\begin{pmatrix}2\\6\\-20\end{pmatrix} = \frac{1}{\sqrt{110}}\begin{pmatrix}1\\3\\-10\end{pmatrix}$$

Upshot $\boxed{B = UDU^{-1}}$

where $D = \begin{pmatrix}-5 & & 0\\ & 5 & \\ 0 & & 5\end{pmatrix}$

$$U = \begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix}$$

Geometric intuition : U are rotations

any symmetrix matrix B can be decomposed as

rescaling along coordinate axes

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "D"] & \mathbb{R}^n \arrow[d, "U"] \\ \arrow[u, "U^{-1}"] \arrow[r, "B"] & \end{tikzcd}$$

inverse rotation

rotation

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n$$

Ex.

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, bend left, "\begin{pmatrix} \frac{5+\sqrt{5}}{2} & 0 \\ 0 & \frac{5-\sqrt{5}}{2} \end{pmatrix}"] & \mathbb{R}^2 \arrow[d, "\text{rotation}"] \\
\mathbb{R}^2 \arrow[u, "\text{inverse rotation}"] \arrow[r, bend left, "\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}"] & \mathbb{R}^2
\end{tikzcd}$$

Which rotation is it in the case of $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$?

$$U = \begin{pmatrix} \frac{1}{d_1} & \frac{1}{d_2} \\ \frac{\frac{1}{2}+\frac{\sqrt{5}}{2}}{d_1} & \frac{\frac{1}{2}-\frac{\sqrt{5}}{2}}{d_2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $\theta = \arccos\left(\frac{1}{d_1}\right) = \arccos\left(\frac{1}{2}\sqrt{10+2\sqrt{5}}\right)$

So rotation by $\arccos\left(\frac{1}{2}\sqrt{10+2\sqrt{5}}\right)$ radians